5.1 The Riesz Representation Theorem

Back into pre-hilbert space.

Lemma. *in a pre-Hilbert (no completeness) space the parallelogram law holds:*

$$\forall x, y \in H \qquad 2(\|x\|^2 + \|y^2\|) = \|x - y\|^2 + \|x + y\|^2$$

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Proof. Just derive from $\langle x-y, x-y\rangle$ and $\langle x+y, x+y\rangle$. $\square$

Theorem. *Conversely, any norm space in which this law holds is a pre-Hilbert space. (i.e. $\exists$ an inner product such that $\|x\|^2 = \langle x, x\rangle$.)*

Proof. Define

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

we also have to prove linearity, but thats not hard. $\square$

Proposition. *If $C \subset H$ is a closed convex subset of a Hilbert Space then $\exists! x^* \in C$ such that*

$$\|x^*\| = \inf_{x \in C} \|x\|$$

Proof. We first define convex. C convex implies that for $x, y \in C$ then $\frac{x+y}{2} \in C$ (a rather weak notion of it, but it works in our case)

Now continue with the proof. By definition $\exists x_n \in C$ such that $\|x_n\|^2 \longrightarrow \inf_{x\in C} \|x\|^2 = I^2$. We claim that $\{x_n\}$ is automatically Cauchy. Given $\epsilon > 0$, $\exists N$ such that

$$n > N \Rightarrow \|x_n\|^2 = \inf_{x\in C} \|x\|^2 + \delta$$

Now, note that $\|x_n + x_m\|^2 = 4\|\frac{x_n+x_m}{2}\|^2$. Then

$$\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2 \le 4I^2 + 4\delta - 4I^2 \le 4\delta, \text{ since } \left\|\frac{x_n + x_m}{2}\right\|^2 \ge I^2$$

So if $n, m \ge N$, then $\|x_n - x_m\| \le 4\delta$ so $\{x_n\}$ is Cauchy $\square$

Theorem. ***Riesz Representation Theorem*** *If H is a hilbert space and $g : H \to \mathbb{C}$ (a functional) is linear and continuous then $\exists y \in H$ such that*

$$g(x) = \langle x, y\rangle, \qquad \forall x \in H$$

First, what does continuous mean? $g : H \to \mathbb{C}$ continuous iff $g^{-1}(O) \subset H$ is open $\forall O \subset \mathbb{C}$ open, which implies that $g^{-1}(B(0,1)) \subset H$ is open. i.e. $\{x \in H | |g(x)| < 1\} \subset H$ is open. (All of this is because g is linear, i.e. you only need continuity around the origin) or simiarly $\{x \in H : |g(x)| \le 1\}$ is closed.

$\{x \in H : |g(x)| < 1\}$ is open implies $\exists \epsilon > 0$ such that $\{x \in H : \|x\| < \epsilon\} \Rightarrow \|g(x)\| < 1$. g is linear, so $x \in H$ $\|x\| \le 1 \Rightarrow \|\frac{\epsilon}{2}x\| < \epsilon$, so $|g(\frac{\epsilon}{2}x)| < 1 < 2$, so

$$g(x) = \left|\frac{2}{\epsilon} g\left(\frac{\epsilon}{2}x\right)\right| \le \frac{4}{\epsilon} = c$$

So $\exists$ a constant c such that

$$\|x\| \le 1 \Rightarrow |g(x)| \le c$$

that is, we have boundedness, $\forall x \in H$, $|g(x)| \leq C\|x\|$. Thus, by the boundedness around the origin,

$$\left|g\left(\frac{x}{\|x\|}\right)\right| \leq C \Rightarrow \left|\frac{1}{\|x\|}g(x)\right| = \frac{1}{\|x\|}g(x), \qquad \text{so } |g(x)| \leq C\|x\|$$

so by continuity of g, $\exists C$ such that $|g(x)| \leq C\|x\|$, $\forall x \in H$.

But it works conversely. To get $\Leftarrow$ direction: If $x_n \to x$ then $x_n - x \to 0$, i.e. given $\epsilon > 0$ $\exists N$ such that $\|x_n - x\| < \epsilon$ if $n \geq N$, then

$$|g(x_n) - g(x)| = |g(x_n - x)| \leq C\|x_n - x\| \to 0$$

So for linear functions, boundedness is the same as continuity,

$$\boxed{|g(x)| \leq C\|x\|}$$

Proof. If $y \in H$ by Cauchy-Schwarz implies

$$|\langle x, y\rangle| \leq \|x\|\|y\| = C\|x\|$$

so this jibes with $|g(x)| \leq C\|x\|$.

Conversely, given $g : H \to \mathbb{C}$ linear and continuous find y such that $g(x) = \langle x, y\rangle$. $g \equiv 0$ implies that $y = 0$. g non-zero

$$g(x') = 1, \qquad x' = \frac{x}{g(x)}$$

Set

$$C_g = \{x \in H : g(x) = 1\}$$

This is closed (inverse image of a single point) and convex, since

$$g\left(\frac{x+y}{2}\right) = \frac{1}{2}g(x) + \frac{1}{2}g(y) = 1$$

This is closed, convex. By the proposition above $\exists! y^* \in C_g$ such that $\|y^*\| = \inf_{x \in C_g} \|x\|$. Then

$$C_g = \{y^* + y : y \in H, g(y) = 0\}$$

$y^* \perp N$, $N = \{x \in H : g(x) = 0\}$, the above can be re-written as $C_g = \{y^* + y : y \in N\}$. So then $y \in N \Rightarrow \langle y, y^*\rangle = 0$, because

$$\|y^* + ty\|^2 = \|y^*\|^2 + t\langle y, y^*\rangle + t\langle y, y^*\rangle + t^2\|y\|^2$$

We claim

$$x \in H \Rightarrow x = sy^* + y, \qquad y \in N, s \in \mathbb{C}$$

Proof:

$$g(x) = 0 \Rightarrow x \in N$$

so then

$$g(x) = s \Rightarrow g\left(\frac{x}{s}\right) = 1 \Rightarrow \frac{x}{s} \in C_g \Rightarrow x = sy^* + y$$

then

$$g(x) = g(sy^* + y) = sg(y^*) + g(y) = s$$

Then $x \in H$ implies $x = g(x)y^* + y, \qquad g(y) = 0$ then

$$\langle x, y^* \rangle = g(x)\langle y^*, y^* \rangle \Rightarrow g(x) = \left\langle x, \frac{y^*}{\|y^*\|^2} \right\rangle$$

$\square$

Example. If $g : L^2(X, \mu) \to \mathbb{C}$ is continuous and linear, then $\exists G \in L^2$ such that

$$g(f) = \int f\overline{G}d\mu$$